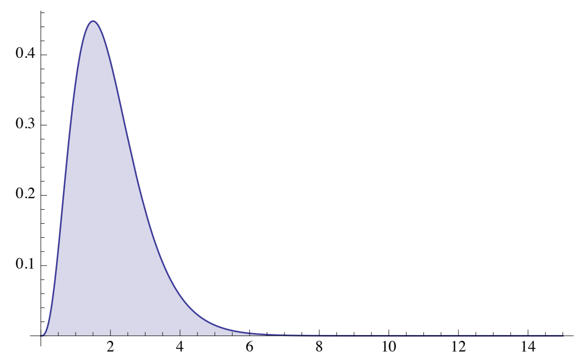
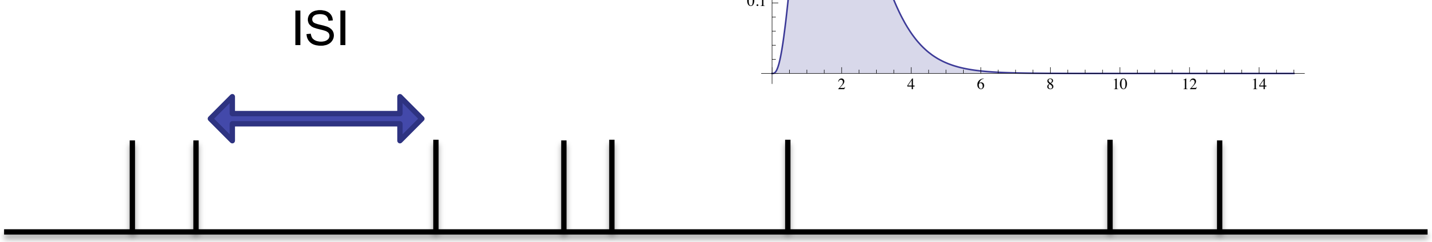
RENEWAL PROCESS

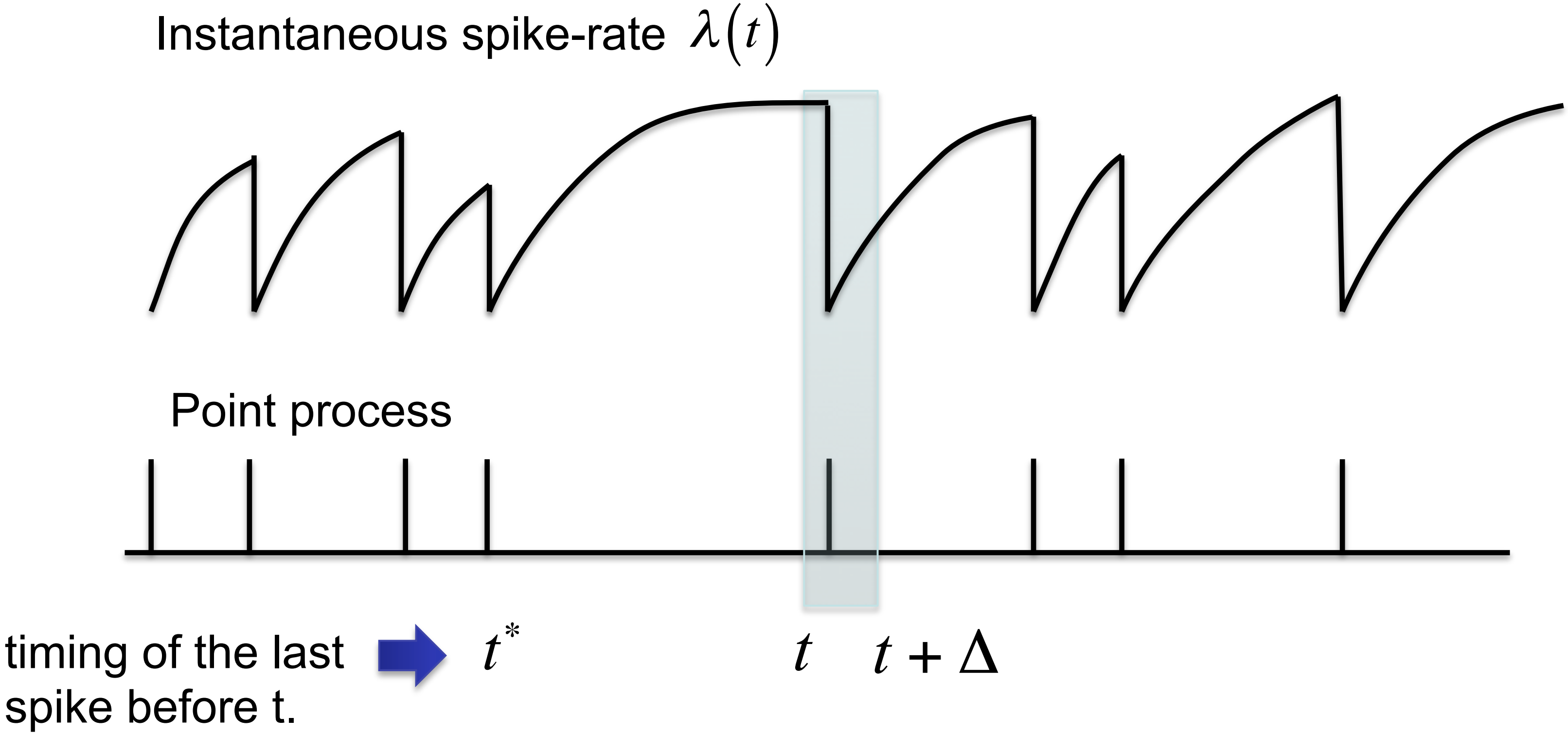
Renewal process

Renewal process: ISIs are independent and identically distributed.





Instantaneous rate of a renewal process



$$P\left(\text{a spike in } [t, t+\Delta)\right) = \lambda\left(t - t^*\right)\Delta + o(\Delta)$$

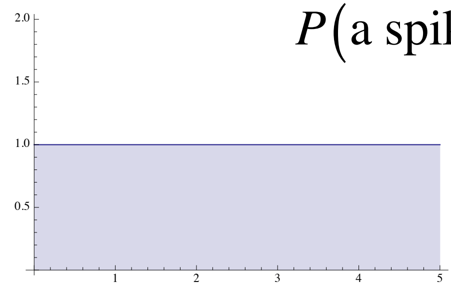
Instantaneous rate depends on the elapsed time from the last spike.

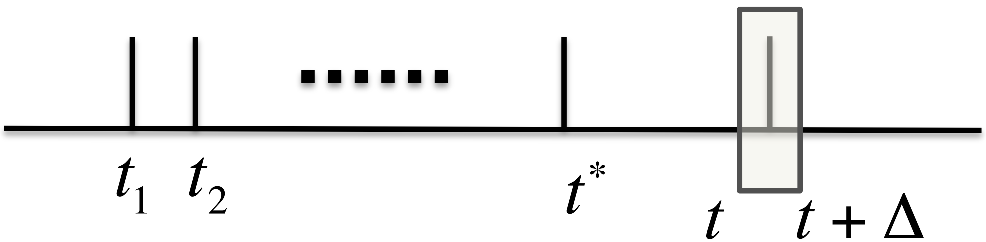
Instantaneous rate and ISI-density

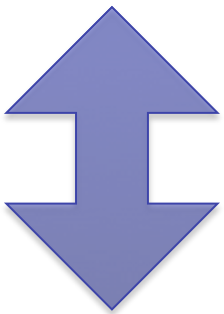
Poisson process

$$P(\text{a spike in } [t, t+\Delta)) = \lambda\Delta + o(\Delta)$$

Constant intensity λ

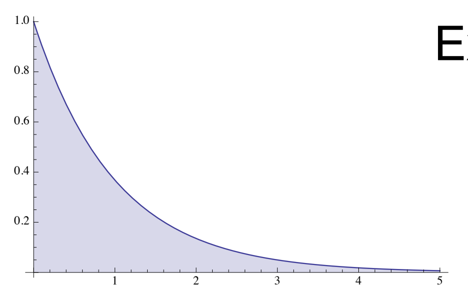






Exponential ISI distribution

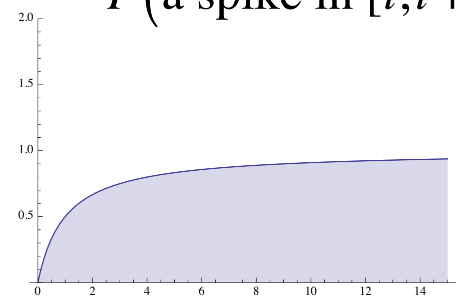
$$f(\tau) = \lambda e^{-\lambda\tau}$$



Instantaneous rate and ISI distribution

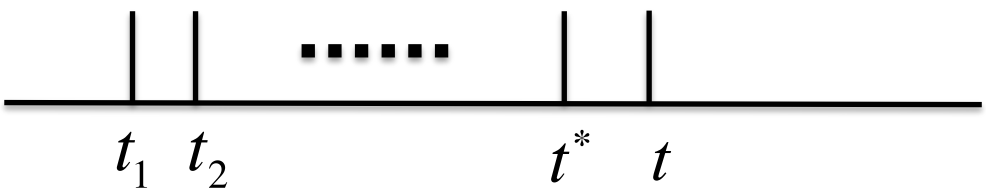
Renewal process

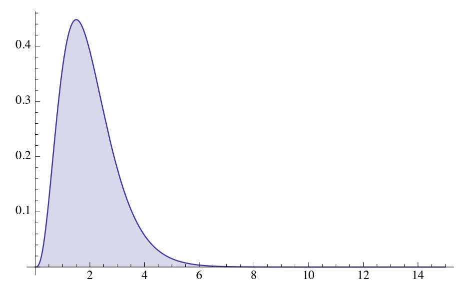
$$P\left(\text{a spike in } [t, t+\Delta)\right) = \lambda\left(t - t^{*}\right)\Delta + o(\Delta)$$



Time-dependent rate

$\lambda(\tau)$





Other ISI distribution

$f(\tau)$

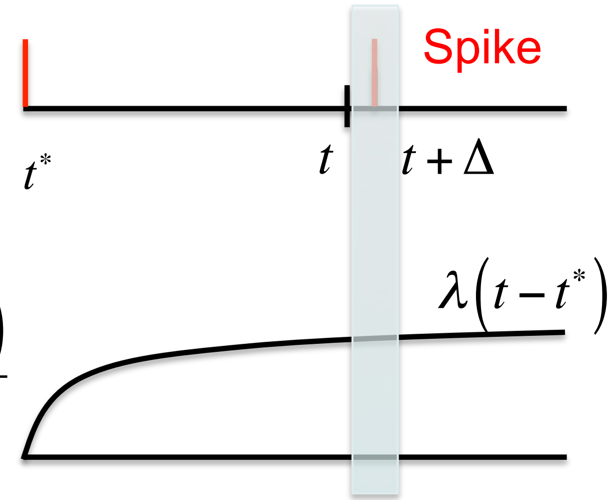
What is the relation between ISI and instantaneous rate for a renewal process?

Conditional intensity function (CIF)

Conditional intensity function (CIF)

Probability that a spike occurs at time t given that no spikes are generated for time t.

$$\lambda\left(t \mid t^*\right) = \lim_{\Delta \to 0} \frac{P\left(\text{a spike at [t,t+}\Delta\text{)} \middle| \text{no spike in } [t^*, \text{t)}\right)}{\Delta}$$



The CIF is an instantaneous spike-rate. It is also called the hazard function, age-specific failure rate, and recovery function.

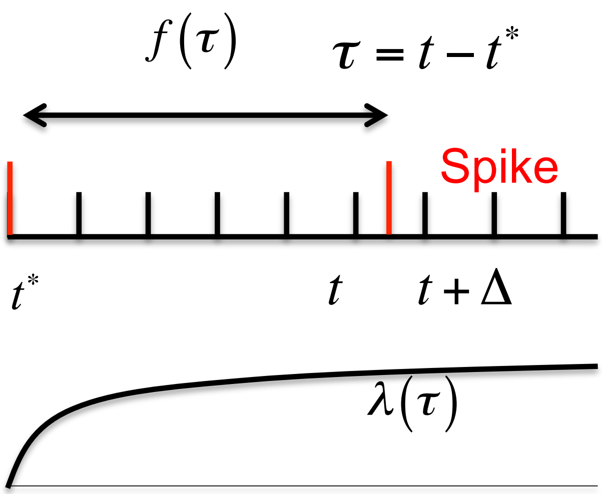
ISI density and CIF

The relation between the ISI density and the conditional intensity function (CIF)

Elapsed time since the last spike: $\tau = t - t^*$

$$\lambda(\tau) = \frac{f(\tau)}{1 - \int_0^\tau f(u)\,du}$$

$$f(\tau) = \lambda(\tau)\exp\left[-\int_0^\tau \lambda(u)\,du\right]$$



Conditional intensity and ISI distribution

Relation between the conditional intensity function and ISI distribution

$$\lambda(\tau) = \lim_{\Delta \to 0} \frac{P(\tau < X \le \tau + \Delta | X > \tau)}{\Delta}$$

X : Random variable for ISI.

$$= \lim_{\Delta \to 0} \frac{P(\tau < X \le \tau + \Delta, X > \tau)}{\Delta} \frac{1}{P(X > \tau)}$$

$$= \lim_{\Delta \to 0} \frac{P(\tau < X \le \tau + \Delta)}{\Delta} \frac{1}{P(X > \tau)}$$

$$= \frac{f(\tau)}{\bar{F}(\tau)}$$

$$= \frac{f(\tau)}{1 - F(\tau)}$$

Homework 2-1

Obtain the relations, $f(\tau) = \lambda(\tau)\exp\left[-\int_0^\tau \lambda(u)\,du\right]$

By rewriting the relation and ISI distribution, we obtain

$$\lambda(\tau) = \frac{-\bar{F}'(\tau)}{\bar{F}(\tau)} = -\frac{d}{dt}\log\bar{F}(\tau)$$

Thus, $\frac{d}{d\tau}\log\bar{F}(\tau) = -\lambda(\tau)$, or $d\log\bar{F}(\tau) = -\lambda(\tau)\,d\tau$

Integrating the both sides of the equality from 0 to τ yields

$$\bar{F}(\tau) = \exp\left[-\int_0^\tau \lambda(u)\,du\right] \qquad \log\bar{F}(\tau) - \log\bar{F}(0) = -\int_0^\tau \lambda(u)\,du$$

$$F(\tau) = 1 - \exp\left[-\int_0^\tau \lambda(u)\,du\right]$$

Derivative of the CDF yields the above expression.

Alternative solution: generalize the derivation done in Homework 1-3.

Example: Gamma process

Gamma process

ISI density $f(\tau) = \frac{1}{\Gamma(\kappa)} \kappa (\kappa\lambda\tau)^{\kappa-1} e^{-\kappa\lambda\tau}$

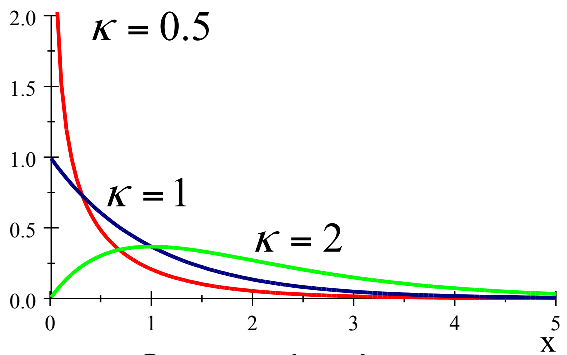
Intensity function $\lambda(\tau) = \frac{f(\tau)}{1 - F(\tau)}$

κ Shape parameter

$\Gamma(\kappa)$ Gamma function

$\Gamma(\kappa) = \int_0^\infty u^{\kappa-1} e^{-u} \, du$

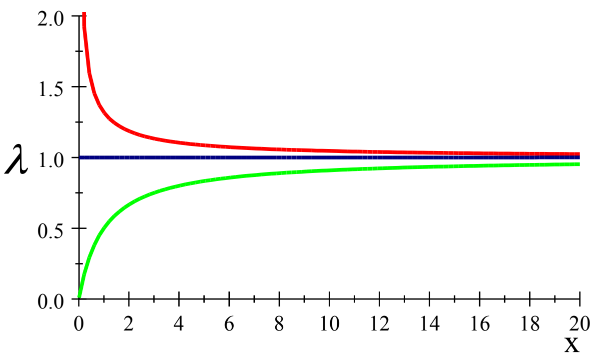
ISI density



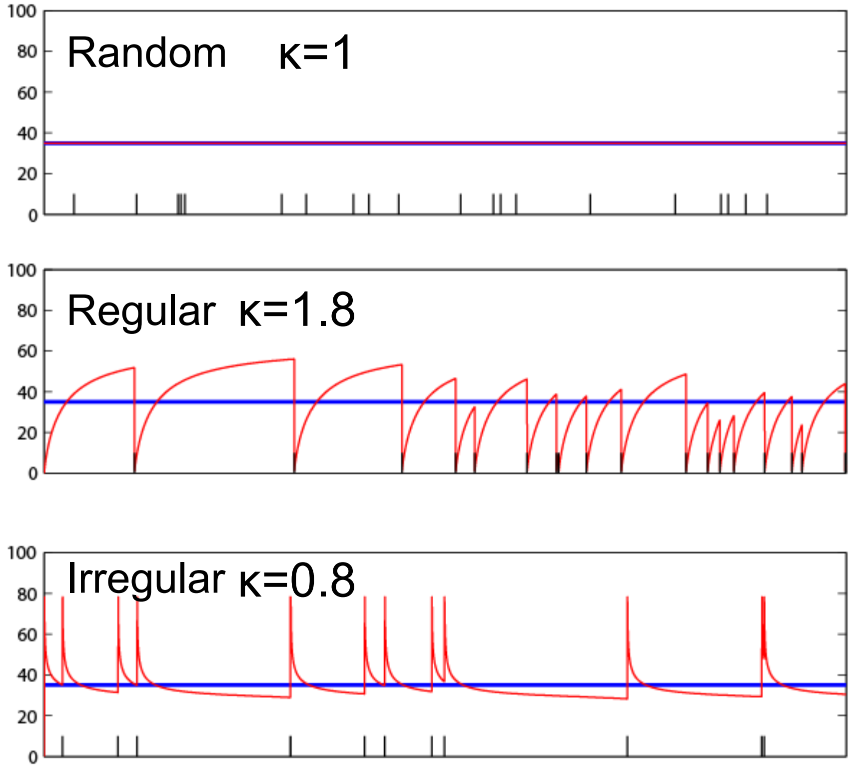
Gamma density

$$f(\tau) = \frac{1}{\Gamma(\kappa)} \kappa (\kappa\lambda\tau)^{\kappa-1} e^{-\kappa\lambda\tau}$$

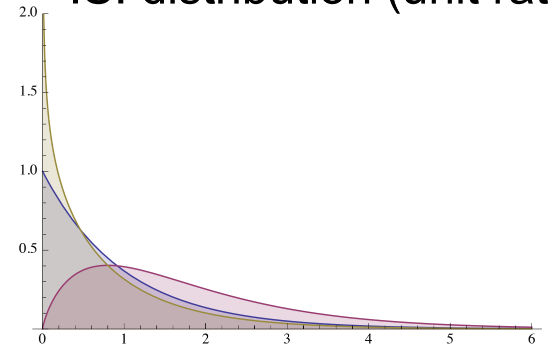
Intensity function



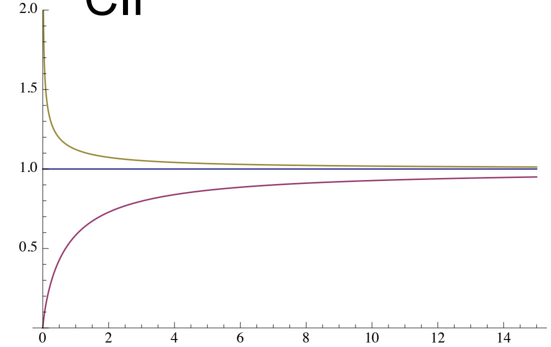
Gamma process



ISI distribution (unit rate)

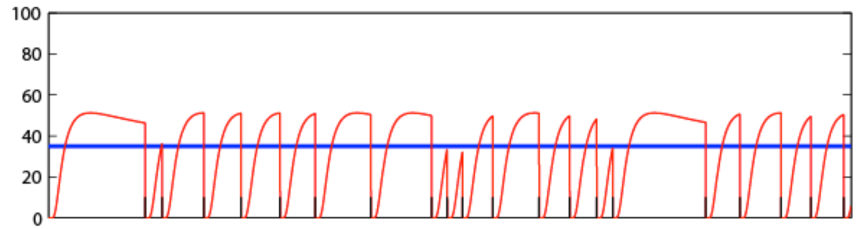


CIF

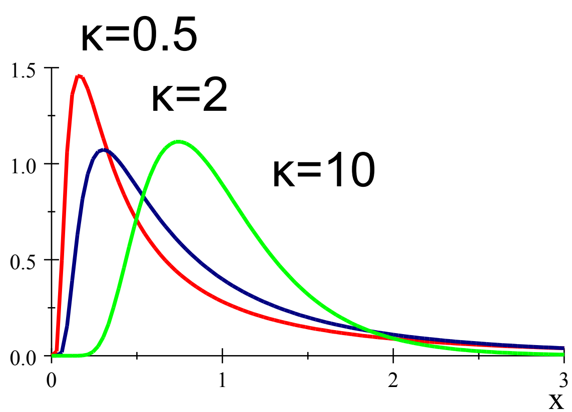


Inverse Gaussian model

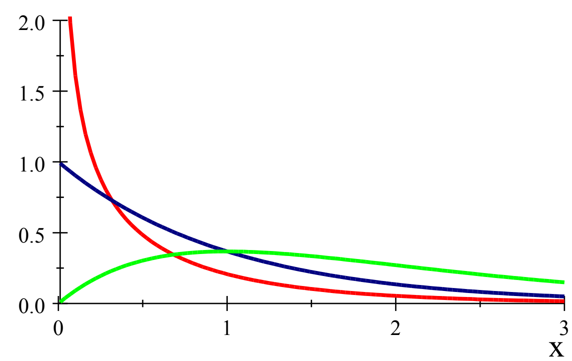
Inverse Gaussian κ=1.8



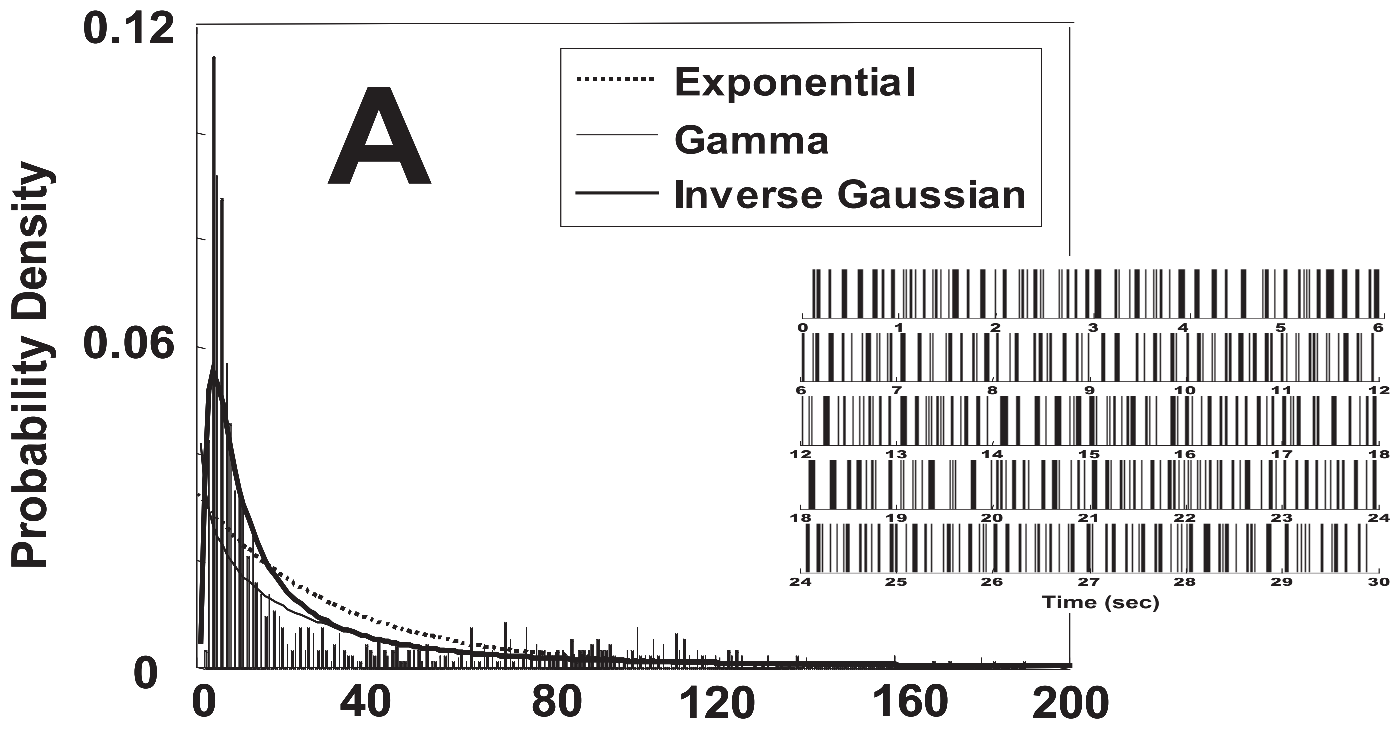
Inverse Gaussian model



Intensity function



Which renewal model for neuronal spikes?



ISI distribution of spike trains from gold fish retinal ganglion cells. Lines are maximum likelihood fits of Poisson, gamma, and inverse Gaussian models density.

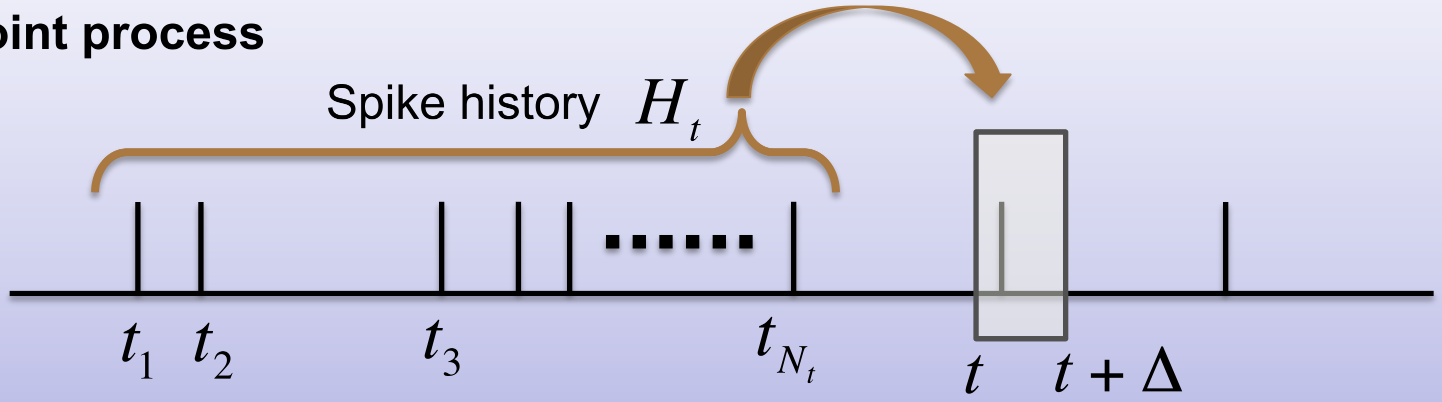
Brown et al. (2004) Computational Neuroscience : A Comprehensive Approach. CRC Press.

The point processes written by the conditional intensity function

POINT PROCESSES

Conditional intensity function

Point process



Definition of a point process

Causal point processes are completely characterized by the conditional intensity function.

Conditional intensity function

$$P\left(\text{a spike in } [t, t+\Delta) \mid H_t\right) = \lambda\left(t \mid H_t\right)\Delta + o\left(\Delta\right)$$

$$P\left(>1 \text{ spikes in } [t, t+\Delta) \mid H_t\right) = o\left(\Delta\right)$$

$$P\left(\text{no spikes in } [t, t+\Delta) \mid H_t\right) = 1 - \lambda\left(t \mid H_t\right)\Delta + o\left(\Delta\right)$$

Spike history: $H_t = \left\{t_1, t_2, \cdots, t_{N_t}\right\}$, where N_t is the number of spikes in (0,t].

CIF of point processes

$$P\left(\text{a spike in } [t, t+\Delta) \mid H_t\right) = \lambda\left(t \mid H_t\right)\Delta + o\left(\Delta\right)$$

Spikes are independent each other = Poisson process

$\lambda\left(t \mid H_t\right) = \lambda$ Homogeneous Poisson process

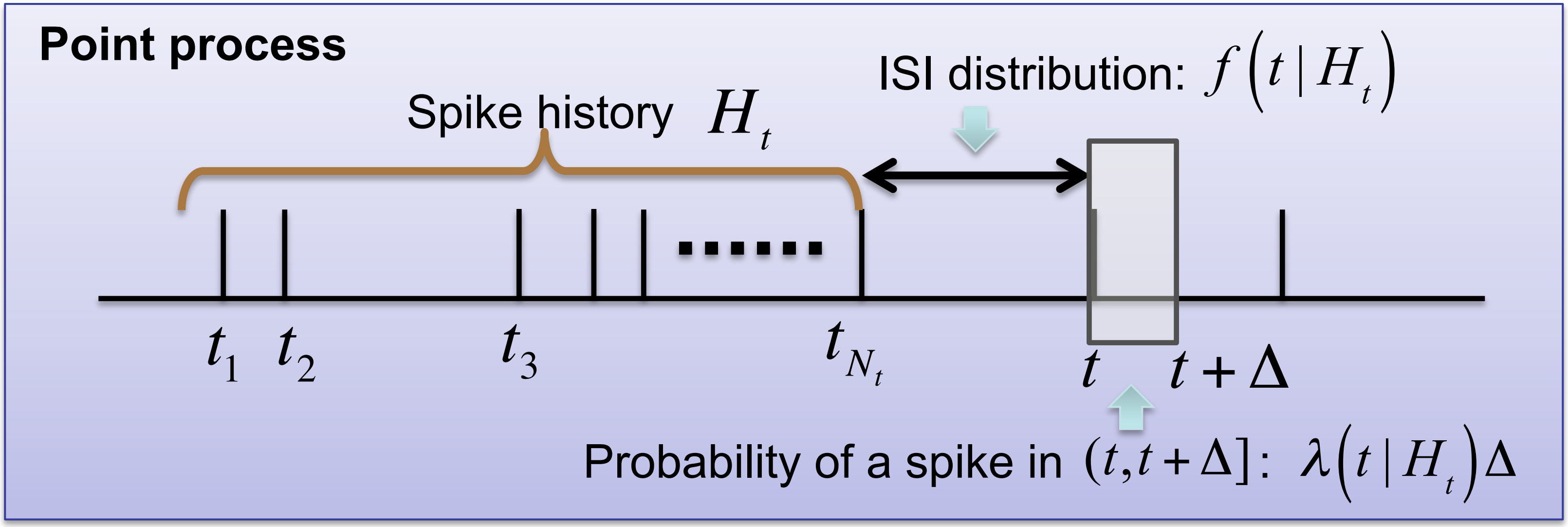
$\lambda\left(t \mid H_t\right) = \lambda\left(t\right)$ Inhomogeneous Poisson process

Spikes are not independent each other = Non-Poisson process

$\lambda\left(t \mid H_t\right) = \lambda\left(t \mid t_{N_t}\right) = \lambda\left(t - t_{N_t}\right)$ Renewal process

time of the last spike before t.

CIF and conditional ISI distribution



Conditional intensity and ISI distribution

$$\lambda(t \mid H_t) = \frac{f(t \mid H_t)}{1 - \int_{t_{N_t}}^{t} f(u \mid H_u)\, du}$$

$$f(t \mid H_t) = \lambda(t \mid H_t) \exp\left\{ -\int_{t_{N_t}}^{t} \lambda(u \mid H_u)\, du \right\}$$

ISI distribution: $f(t \mid H_t) \quad t > t_{N_t}$
The density of the next spike at time t, given given the spike history.

Likelihood function of a point process

Likelihood function

$$p\left(t_1,t_2,\ldots,t_n \cap N_T = n\right) = \prod_{i=1}^{n} \lambda\left(t_i \mid H_{t_i}\right) \exp\left[-\int_0^T \lambda\left(u \mid H_u\right) du\right]$$

Proof

$$p\left(t_1,t_2,\ldots,t_n \cap N_T = n\right)\Delta^n = f\left(t_1\right)\Delta \prod_{i=2}^{n} f\left(t_i \mid H_{t_i}\right)\Delta \cdot P\left(t_n > T \mid H_{t_n}\right)$$

$$= \prod_{i=1}^{n} \lambda\left(t_i \mid H_{t_i}\right)\Delta \exp\left[-\int_0^T \lambda\left(u \mid H_u\right) du\right]$$

Here the ISI distribution is $f\left(t_i \mid H_{t_i}\right) = \lambda\left(t_i \mid H_{t_i}\right) \exp\left[-\int_{t_{i-1}}^{t_i} \lambda\left(u \mid H_u\right) du\right]$

In application to simulating a point process

TIME-RESCALING THEOREM

Time-rescaling theorem

By the time-rescaling, any (causal) point process can be transformed into a Poisson point process with an unit rate.

Time-rescaling theorem

Let $0 < t_1 < t_2 < \cdots < t_n < T$ be a realization from a point process with a conditional intensity function $\lambda(t \mid H_t)$ for $t \in (0,T]$.

Then define the transformation $\Lambda(t_i) = \int_0^{t_i} \lambda(u \mid H_u)\, du$ for $i = 1, \cdots, n$.

Then the $\Lambda(t_i)$ $(i = 1, \cdots, n)$ are a Poisson process with unit rate.

Brown et al. (2001), Daley and Vere-Jones (1988).

Time rescaling theorem is used to

1. Simulate a (causal) point process.
2. Assess goodness-of-fit of the point process model to the data.

Example

Inhomogeneous Poisson process

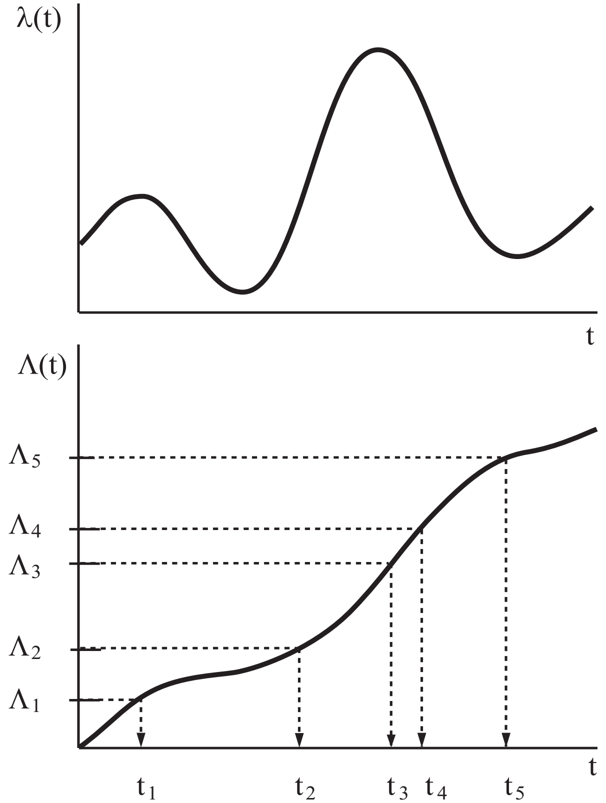
Time-varying instantaneous rate $\lambda(t)$

Time-rescaling $\Lambda(t) = \int_0^t \lambda(u)\,du$

ISI in a rescaled time axis follows an unit exponential distribution.

$$\tau_i = \Lambda(t_i) - \Lambda(t_{i-1}) = \int_{t_{i-1}}^{t_i} \lambda(u)\,du$$

$\tau_i \sim g(\tau)$ where $g(\tau) = e^{-\tau}$



ISI density in the original time scale

$$f(t_i \mid t_{i-1}) = \left|\frac{d\tau_i}{dt_i}\right| g(\tau_i) = \lambda(t_i)\exp\left[-\int_{t_{i-1}}^{t_i} \lambda(u)\,du\right]$$

Proof of time-rescaling theorem

Let the rescaled ISI be $\tau_i = \Lambda(t_i) - \Lambda(t_{i-1})$, where $\Lambda_i = \int_0^{t_i} \lambda(u \mid H_u)\,du.$

We also define $\tau_T = \int_{t_n}^{T} \lambda(u \mid H_u)\,du.$

We prove that the rescaled ISIs are iid exponential random variables:

$$p(\tau_1, \tau_2, \ldots, \tau_n \cap \tau_{n+1} > \tau_T) = \left(\prod_{i=1}^{n} \exp[-\tau_i] \right) \exp(-\tau_T)$$

By change of variables, we obtain the relation:

$$p(\tau_1, \tau_2, \ldots, \tau_n \cap \tau_{n+1} > \tau_T) = p(\Lambda_1, \Lambda_2, \ldots, \Lambda_n \cap N_T = n)$$

$$= \prod_{i=1}^{n} \left| \frac{dt_i}{d\Lambda_i} \right| p(t_1, t_2, \ldots, t_n \cap N_T = n)$$

$$= \exp\left[-\int_0^T \lambda(u \mid H_u)\,du \right]$$

$$= e^{-\tau_1} e^{-\tau_2} \cdots e^{-\tau_n} \cdot e^{-\tau_T}$$

$$c.f. \left| \frac{d\Lambda(t_i)}{dt_i} \right| = \lambda(t_i \mid H_{t_i})$$

Here we used the density of the spike-timing in original axis as

$$p(t_1, t_2, \ldots, t_n \cap N_T = n) = \prod_{i=1}^{n} \lambda(t_i \mid H_{t_i}) \exp\left[-\int_0^T \lambda(u \mid H_u)\,du \right]$$

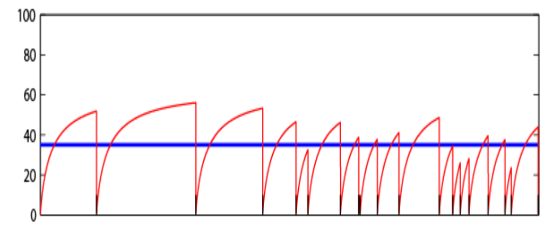
Methods for simulating a point process

1. Method based on the instantaneous rate

At every steps, compute the conditional rate $\lambda(t)$

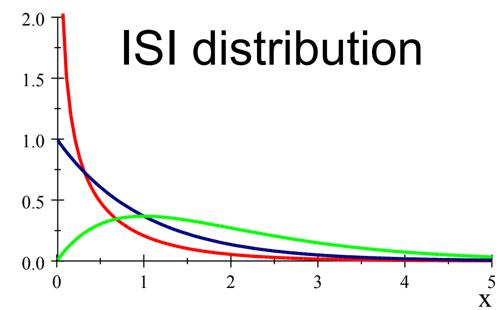
Approximate the spike occurrence by a Bernoulli process. a spike $\lambda(t)\Delta$ no spike $1-\lambda(t)\Delta$

CIF



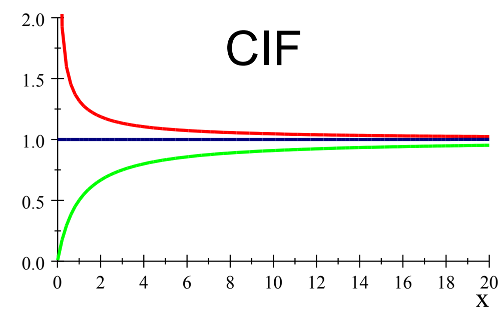
2. Method based on ISI distribution

Generate ISIs using the inverse function method.



3. Method based on the time-rescaling theorem

Generate an exponential random variable.
Integrate the CIF from 0 to t until it reaches the r.v.



Algotihtm based on time-rescaling theorem

Constructing a point process via time-rescaling theorem

0. Let $i = 1$ and $t_0 = 0$.

1. Generate an exponential random variable via the inverse function method.

   $\zeta = -\log \xi$, where ξ is uniform random variable in [0,1].

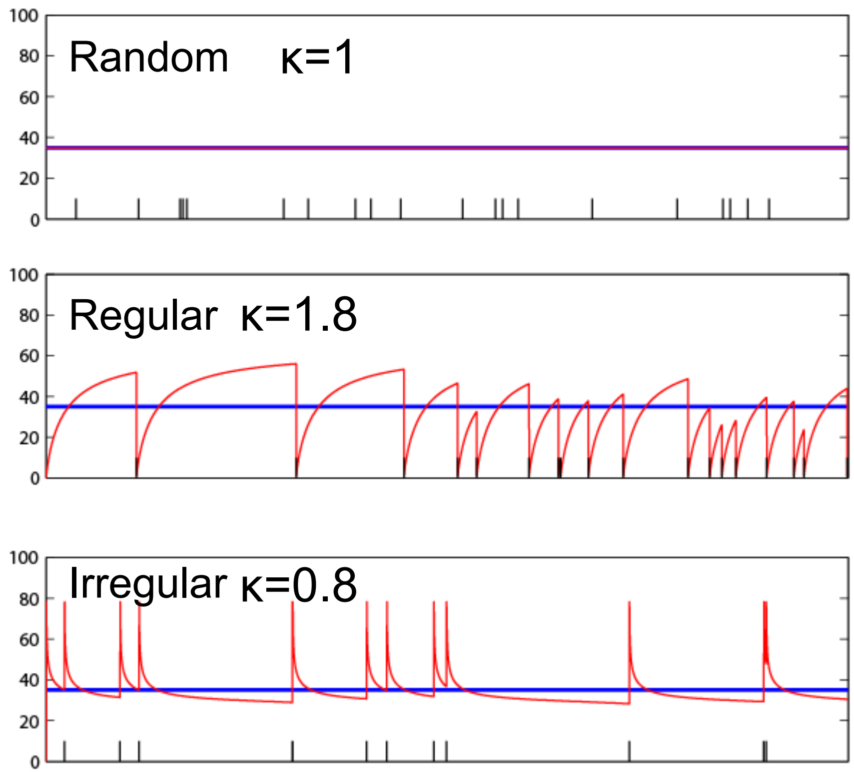
2. Integrate the conditional intensity function until it satisfies

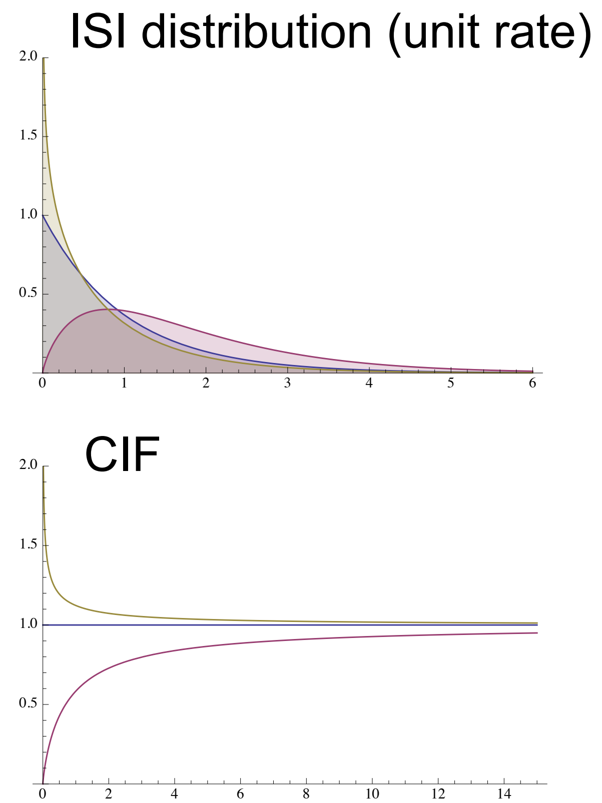
$$\zeta = \int_{t_{i-1}}^{\eta} \lambda(u \mid H_u) du$$

3. Let the i-th spike time as $t_i = \eta$.

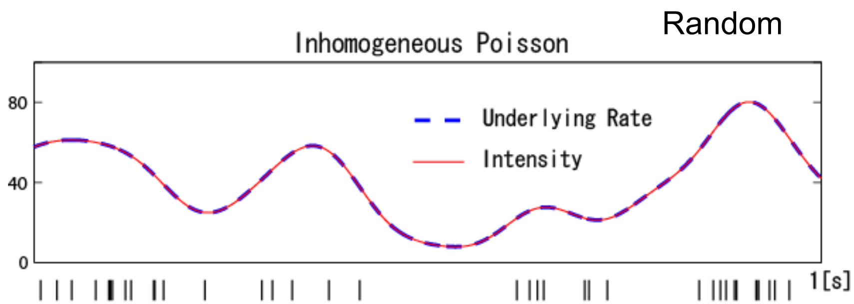
   Update $i \leftarrow i+1$, then repeat 1-3 until the spike time becomes larger than T.

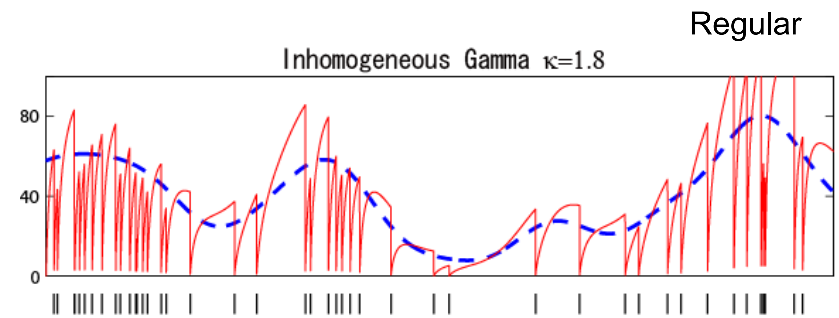
Example: Gamma process

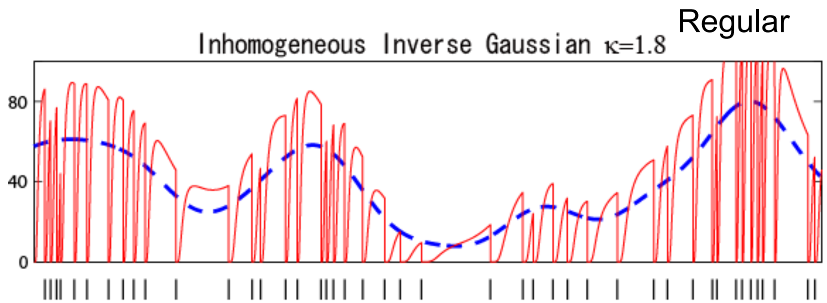


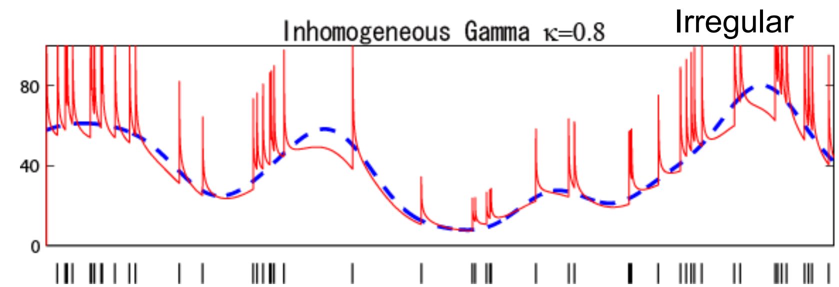


Examples of time-dependent renewal processes

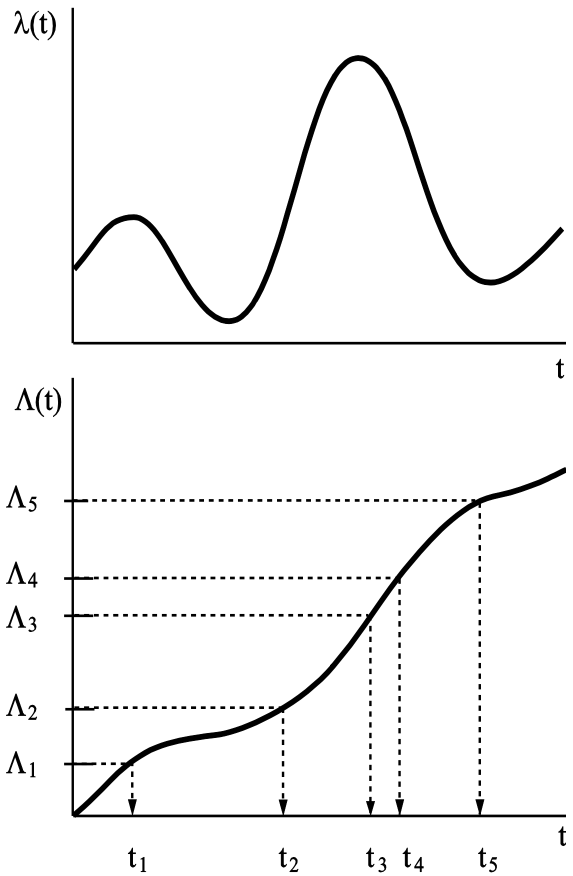








Construction of a time-dependent renewal process



(Time-rescaling)

$$\Lambda(t) = \int_0^t \lambda(u)\,du$$

(Rescaled ISI)

$$\tau\left(t_i \mid t_{i-1}, \lambda_{t_{i-1}:t_i}\right) \equiv \Lambda(t_i) - \Lambda(t_{i-1})$$

(Distribution of rescaled ISI)

$$g(\tau) = \frac{1}{\Gamma(\kappa)} \kappa (\kappa\tau)^{\kappa-1} e^{-\kappa\tau}$$

(Distribution of ISI)

$$p\left(t_i \mid t_{i-1}, \lambda_{t_{i-1}:t_i}\right) = \left|\frac{d\tau}{dt_i}\right| g\left(\tau \mid \lambda_{t_{i-1}:t_i}\right)$$

(Intensity)

$$r\left(t_i \mid t_{i-1}, \lambda_{t_{i-1}:t_i}\right) = \frac{p\left(t_i \mid t_{i-1}, \lambda_{t_{i-1}:t_i}\right)}{1 - \int_{t_{i-1}}^{t_i} p\left(t_i \mid t_{i-1}, \lambda_{t_{i-1}:t_i}\right) dt_i}$$

In application to assessment of model goodness-of-fit

TIME-RESCALING THEOREM

Quantile-Quantile plot

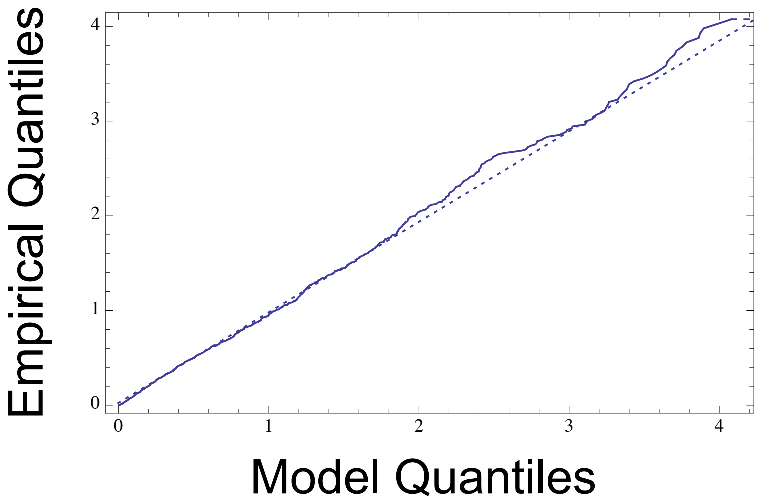
Quantile-Quantile plot

A graphical method for comparing two probability distributions by plotting their quantiles against each other.

Given the two CDFs, $F(x)$ $G(y)$,

the Q-Q plot is a plot given by $\left(F^{-1}(q), G^{-1}(q)\right)$ for $0 \leq q \leq 1$.

Q-Q plot can be used to compare two empirical distributions, two model (parametric) distributions, or an empirical distribution against a model distribution.



Q-Q plot of an unit exponential distribution against empirical distribution of 1000 samples from the unit exponential distribution.

Q-Q plot for assessing point process models

Q-Q plot for assessing a point process model.

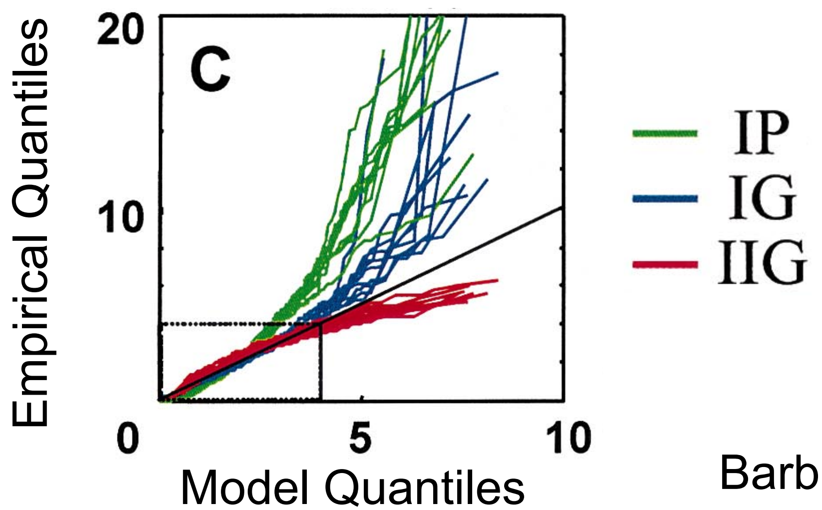
Comparison of the rescaled ISIs and an exponential distribution.

$F(\tau) = 1 - e^{-\tau}$ An exponential distribution function.

$G(\tau)$ An empirical CDF of the rescaled ISIs.

For n ISIs, construct uniform ticks in [0,1]: $b_k = \frac{k - 0.5}{n}$ for $k = 1, \cdots, n.$

Plot $\left(F^{-1}(b_k), G^{-1}(b_k) \right)$ for $k = 1, \cdots, n.$



Barbieri et al. J. Neurosci Methods 2001.

Kolmogorov-Smirnov (K-S) test

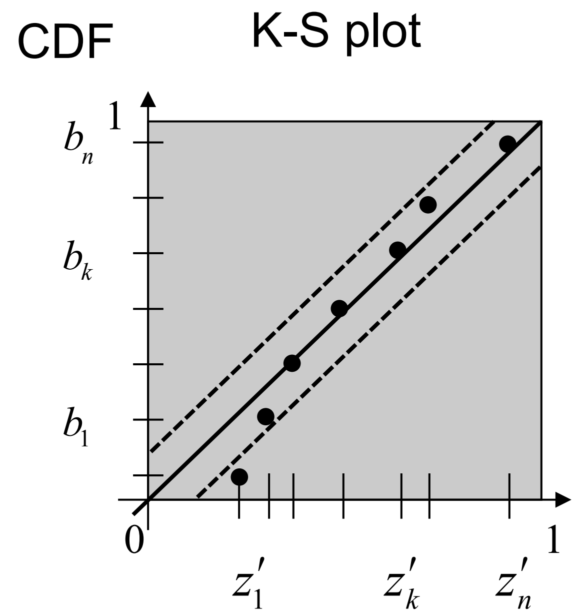
CDF of the rescaled ISIs

$\tau_i \quad i = 1, 2, \cdots, n$

re-scaled ISI that obeys a unit exponential distribution.

$$z_i = 1 - \exp(-\tau_i)$$

z_i obeys a uniform distribution.



K-S plot

Re-order z_i from smallest to largest:

$z_i \rightarrow z'_k$

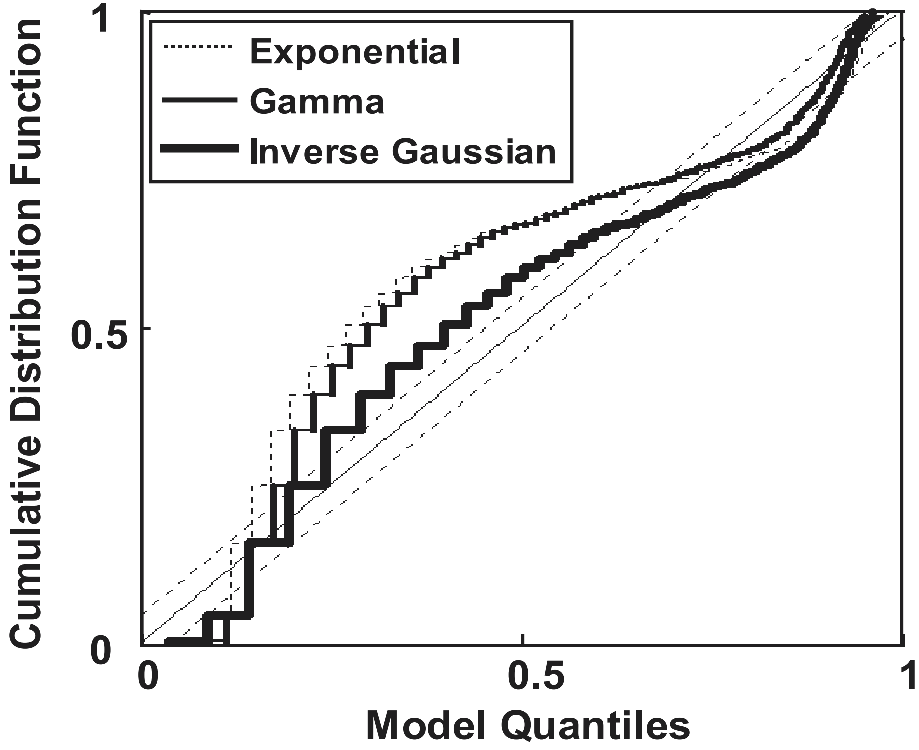
For n ISIs, construct n uniform ticks in [0,1]: $b_k = \frac{k - 0.5}{n}$ for $k = 1, \cdots, n$.

Plot (z_k, b_k) for $k = 1, \cdots, n$.

K-S test 95 confidence bound are well approximated as $b_k \pm 1.36 / n^{1/2}$.

Brown et. al. Neural Comput. 2001
Barbieri et. al. J. Neurosci. Methods 2001

Examples of K-S plots



K-S plots for the fits of (homogeneous) Poisson, gamma, and inverse Gaussian models fitted to spike trains from gold fish retinal ganglion cells.

Brown et al. (2004) Computational Neuroscience : A Comprehensive Approach. CRC Press.

What we learned

1 • **Renewal process: Conditional intensity function** and **ISI density**.

2 • **Point process written by CIF**: conditional ISI distribution, and their likelihood.

3 • **Time-rescaling theorem** (Example of an inhomogeneous Poisson and a proof).

4 • **How to simulate a point process via the time-rescaling theorem.** (Examples of inhomogeneous renewal processes).

5 • **How to assess a point process model** via the time-rescaling theorem. (Q-Q plot and K-S test)

Tomorrow, we will learn

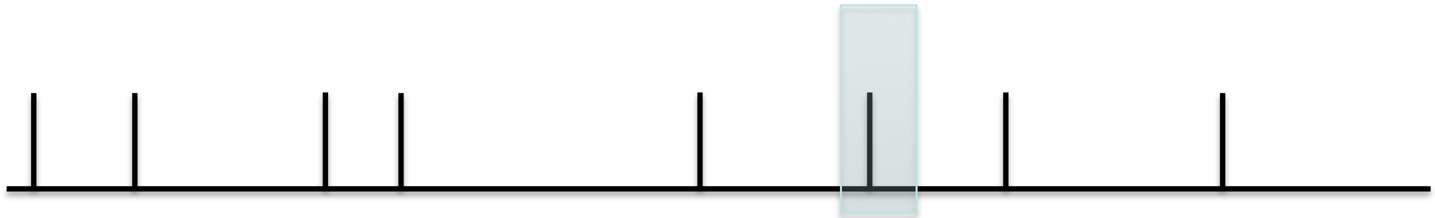
1
- **Exponential family of distributions and generalized liner model.**

2
- **Discrete-time likelihood of conditional Bernoulli and Poisson distributions**.

3
- **GLM framework for a continuous-time point process.**

4
- Review of a paper by Truccolo et al. Focused on methods for model validation**.**

From the feedback of the last lecture



$t \qquad t+\Delta$

$$P\left(\text{a spike in } [t, t+\Delta)\right) = \lambda\Delta + o(\Delta)$$

$$P\left(>1 \text{ spikes in } [t, t+\Delta)\right) = o(\Delta)$$

$$P\left(\text{no spikes in } [t, t+\Delta)\right) = 1 - \lambda\Delta + o(\Delta)$$

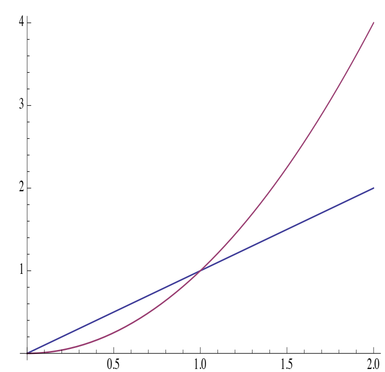
What is $o(\Delta)$?

$$\lim_{\Delta \to +0} \frac{o(\Delta)}{\Delta} = 0$$

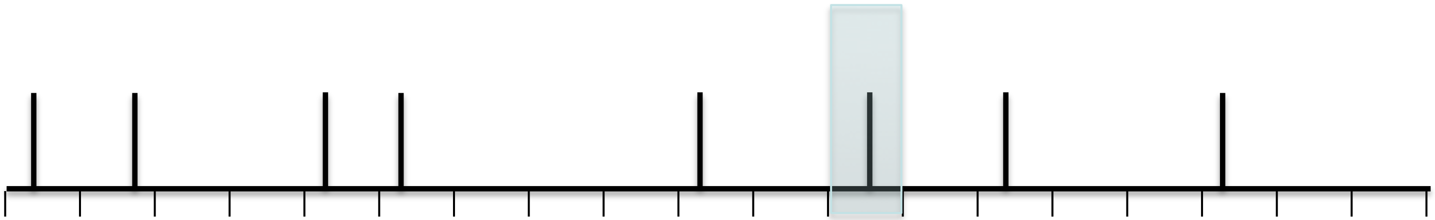
Example

$$o(\Delta) = \Delta^2$$

$$\lim_{\Delta \to +0} \frac{\Delta^2}{\Delta} = \lim_{\Delta \to +0} \Delta = 0$$



From the feedback of the last lecture



If you do not have $o(\Delta)$, namely $o(\Delta) = 0$

$$P(\text{a spike in } [t, t+\Delta)) = \lambda\Delta$$

$$P(>1 \text{ spikes in } [t, t+\Delta)) = 0$$

$$P(\text{no spikes in } [t, t+\Delta)) = 1 - \lambda\Delta$$

This is a Bernoulli process.

From the feedback of the last lecture

We now recall that the number of spike count in a certain interval follows the Poisson distribution.

The spike count in a small bin is then approximated as

$$P\left(N_\Delta = n\right) = \frac{\left(\lambda\Delta\right)^n}{n!} e^{-\lambda\Delta} = \frac{\left(\lambda\Delta\right)^n}{n!}\left[1 - \lambda\Delta + \frac{1}{2}\left(\lambda\Delta\right)^2 + \cdots\right]$$

In particular,

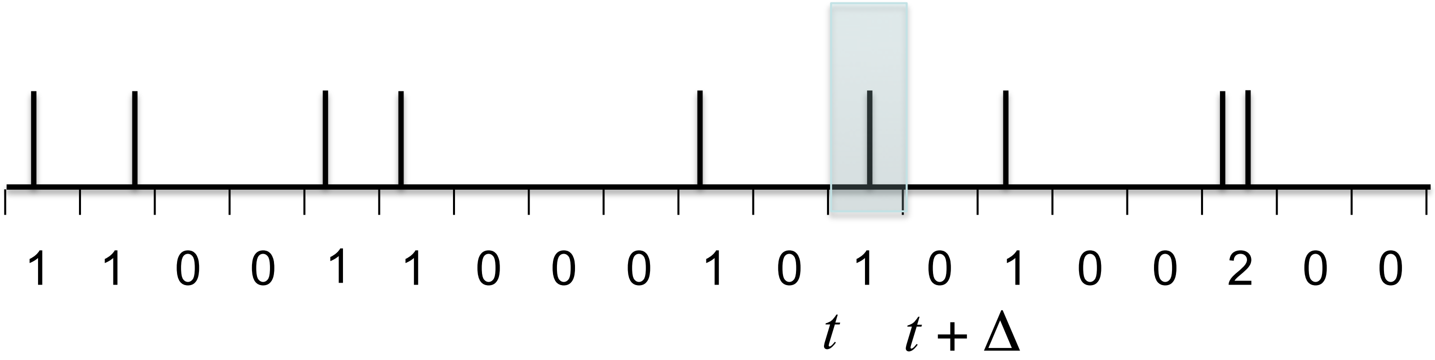
$$P\left(N_\Delta = 0\right) = 1\left[1 - \lambda\Delta + \frac{1}{2}\left(\lambda\Delta\right)^2 + \cdots\right] = 1 - \lambda\Delta + o\left(\Delta\right)$$

$$P\left(N_\Delta = 1\right) = \lambda\Delta\left[1 - \lambda\Delta + \frac{1}{2}\left(\lambda\Delta\right)^2 + \cdots\right] = \lambda\Delta + o\left(\Delta\right)$$

$$P\left(N_\Delta = 2\right) = \left(\lambda\Delta\right)^2\left[1 - \lambda\Delta + \frac{1}{2}\left(\lambda\Delta\right)^2 + \cdots\right] = o\left(\Delta\right)$$

The probability of having a spike/no spike is an approximation of the Poisson count distribution for a small time bin.

From the feedback of the last lecture



$$P(\text{a spike in } [t, t+\Delta)) = \lambda\Delta + o(\Delta)$$

$$P(>1 \text{ spikes in } [t, t+\Delta)) = o(\Delta)$$

$$P(\text{no spikes in } [t, t+\Delta)) = 1 - \lambda\Delta + o(\Delta)$$

This is discrete-time representation of a Poisson process.